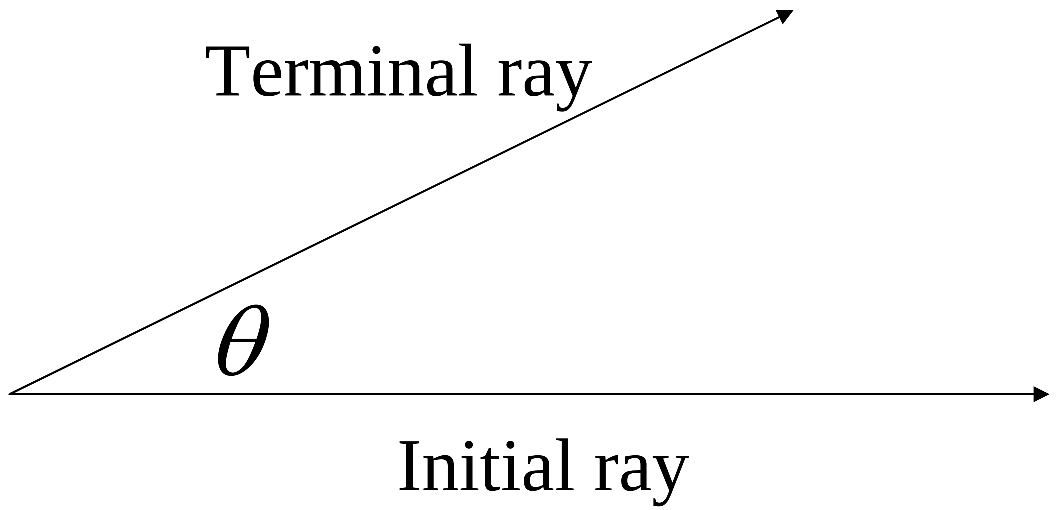
Trigonometric Functions

12

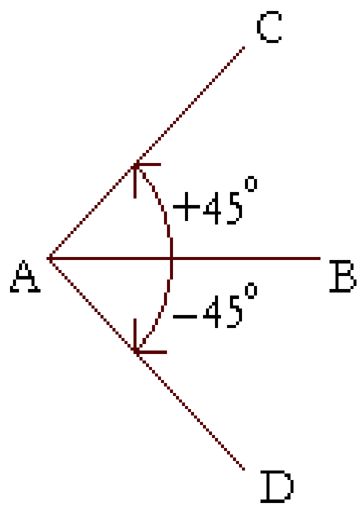
- Measurement of Angles
- Trigonometric Functions
- Differentiation of Trigonometric Functions
- Integration of Trigonometric Functions

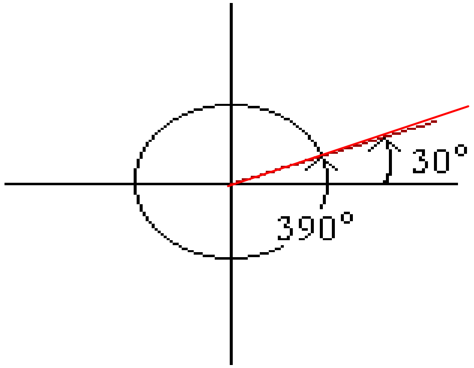
12.1 Measurement of Angles

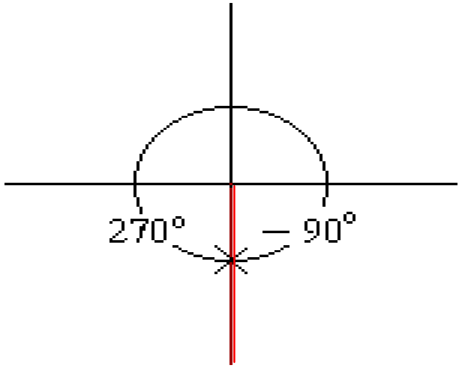
An angle consists of two rays that intersect at a common end point.



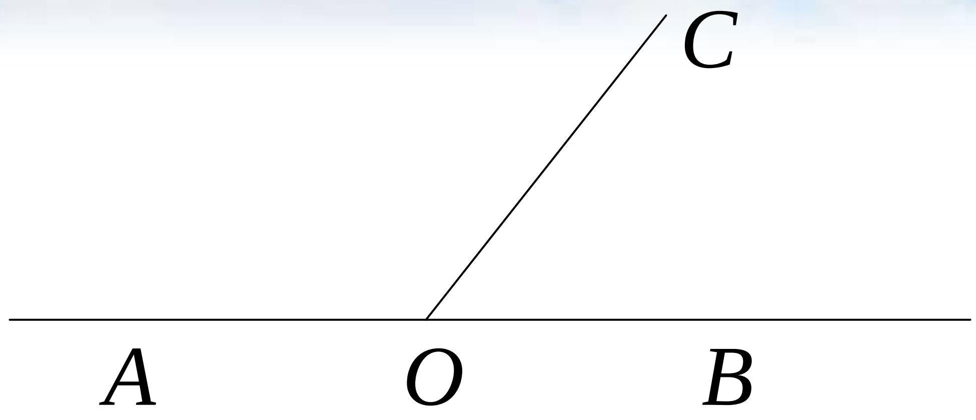
Positive and negative angles



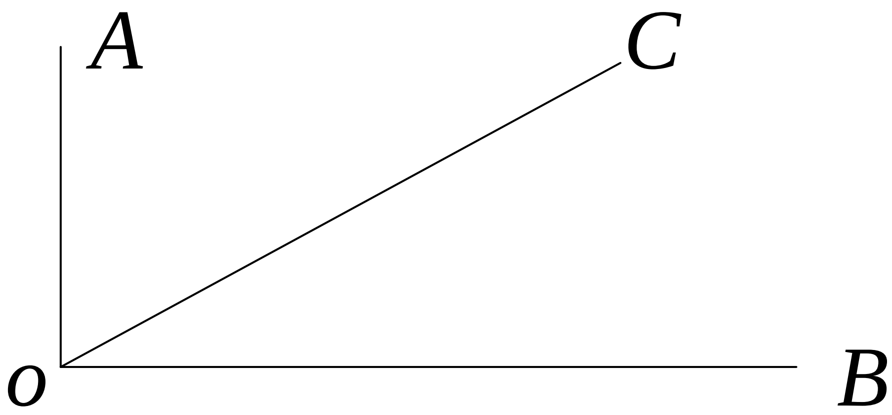




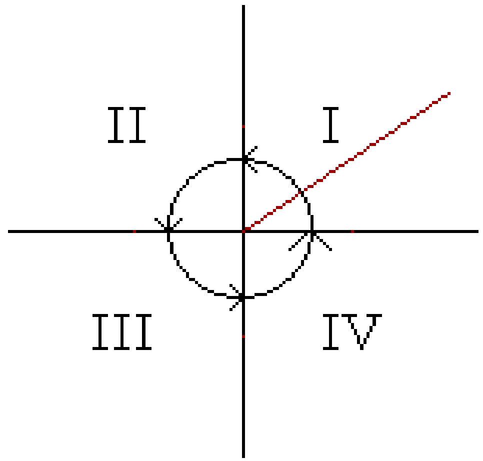
Supplementary angles



Complementary angles



Quadrants



Converting degrees to Radians

$f(x) = \frac{\pi}{180} x$ radians, where x is the number of degrees and $f(x)$ is the number of radians.

EXAMPLE 1 Convert each angle to radian measure:

a. 30° **b.** 45° **c.** 300° **d.** 450° **e.** −240°

Solution Using Formula (3), we have

a. $f(30) = \frac{\pi}{180}(30)$ radians, or $\frac{\pi}{6}$ radians

b. $f(45) = \frac{\pi}{180}(45)$ radians, or $\frac{\pi}{4}$ radians

c. $f(300) = \frac{\pi}{180}(300)$ radians, or $\frac{5\pi}{3}$ radians

d. $f(450) = \frac{\pi}{180}(450)$ radians, or $\frac{5\pi}{2}$ radians

e. $f(-240) = \frac{\pi}{180}(-240)$ radians, or $-\frac{4\pi}{3}$ radians

Converting Radians to degrees

$g(x) = \frac{180}{\pi} x$ degrees, where x is the number of radians and $g(x)$ is the number of degrees.

The radian and degree measures of the common angles:

Degrees	0^0	30^0	45^0	60^0	90^0
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

Example

Convert 315^{0} to radian measure.

Solution

$$f(315) = \frac{\pi}{180} \cdot 315 = \frac{7\pi}{4} \text{ radians}$$

Example

Convert $-\frac{2\pi}{3}$ to degree measure.

Solution

$$g\left(-\frac{2\pi}{3}\right) = \frac{180}{\pi} \cdot \left(-\frac{2\pi}{3}\right) = -120^{0}$$

EXAMPLE 2 Convert each angle to degree measure:

a. $\frac{\pi}{2}$ radians **b.** $\frac{5\pi}{4}$ radians **c.** $-\frac{3\pi}{4}$ radians **d.** $\frac{7\pi}{2}$ radians

Solution Using Formula (4), we have

a. $g\left(\frac{\pi}{2}\right) = \frac{180}{\pi}\left(\frac{\pi}{2}\right)$ degrees, or 90 degrees

b. $g\left(\frac{5\pi}{4}\right) = \frac{180}{\pi}\left(\frac{5\pi}{4}\right)$ degrees, or 225 degrees

c. $g\left(-\frac{3\pi}{4}\right) = \frac{180}{\pi}\left(-\frac{3\pi}{4}\right)$ degrees, or -135 degrees

d. $g\left(\frac{7\pi}{2}\right) = \frac{180}{\pi}\left(\frac{7\pi}{2}\right)$ degrees, or 630 degrees ■

12.2 Trigonometric Functions

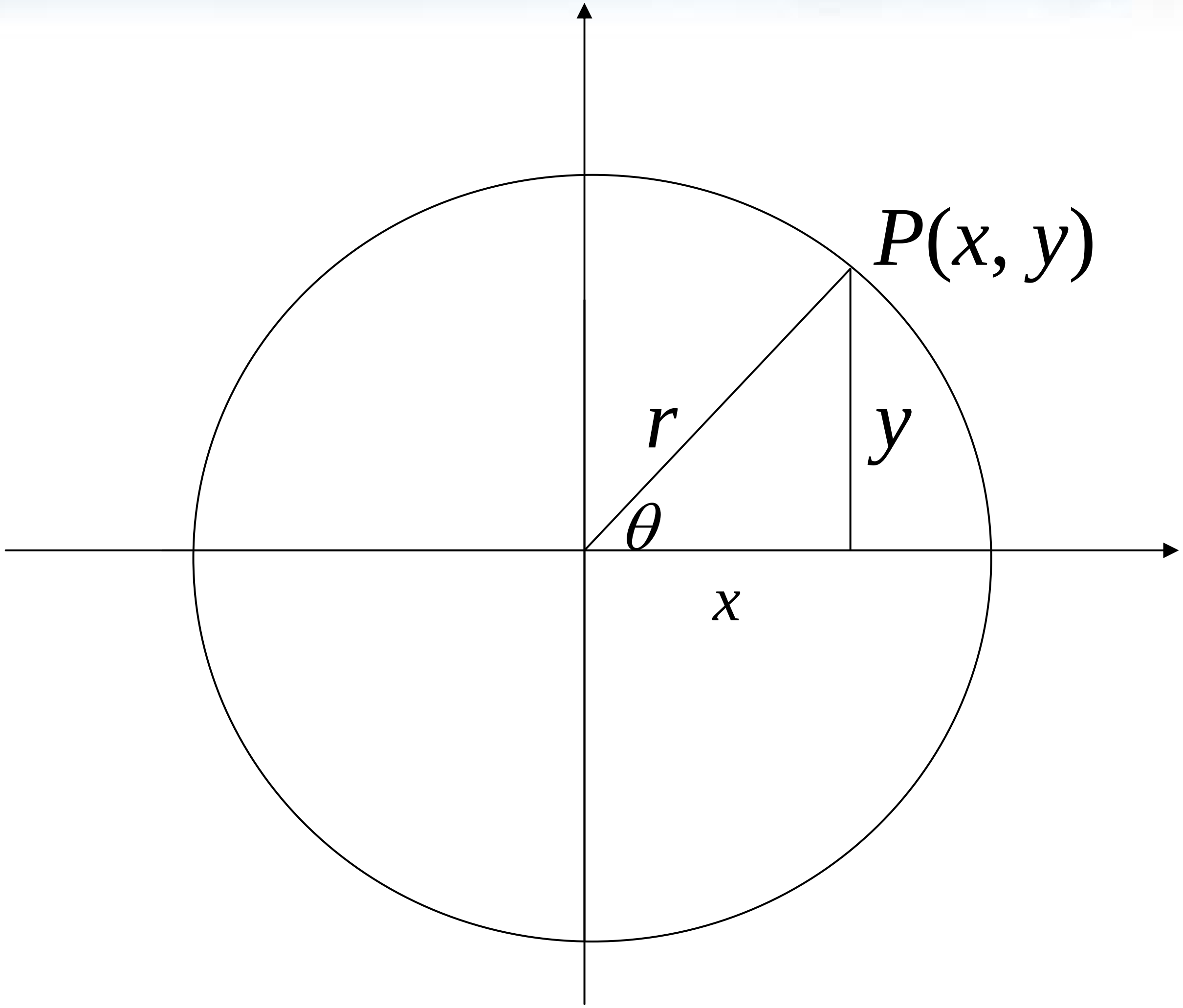
If P is a point on the unit circle and the coordinates of P are (x, y), then

$$\sin\theta = y \qquad \cos\theta = x$$

$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{x}{y} = \frac{\cos\theta}{\sin\theta}$$

$$\sec\theta = \frac{1}{x} = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{y} = \frac{1}{\sin\theta}$$

The Unit Circle and the Point $P\ (x, y)$



EXAMPLE 1 Evaluate

a. $\sin \frac{7\pi}{2}$ **b.** $\cos 5\pi$ **c.** $\sin\left(-\frac{5\pi}{2}\right)$ **d.** $\cos\left(-\frac{11\pi}{4}\right)$

Solution

a. Using (5) and Table 2, we find

$$\sin\left(\frac{7\pi}{2}\right) = \sin\left(2\pi + \frac{3\pi}{2}\right) = \sin\frac{3\pi}{2} = -1$$

b. Using (5) and Table 2, we have

$$\cos 5\pi = \cos(4\pi + \pi) = \cos \pi = -1$$

c. Using (5), (6), and Table 2, we have

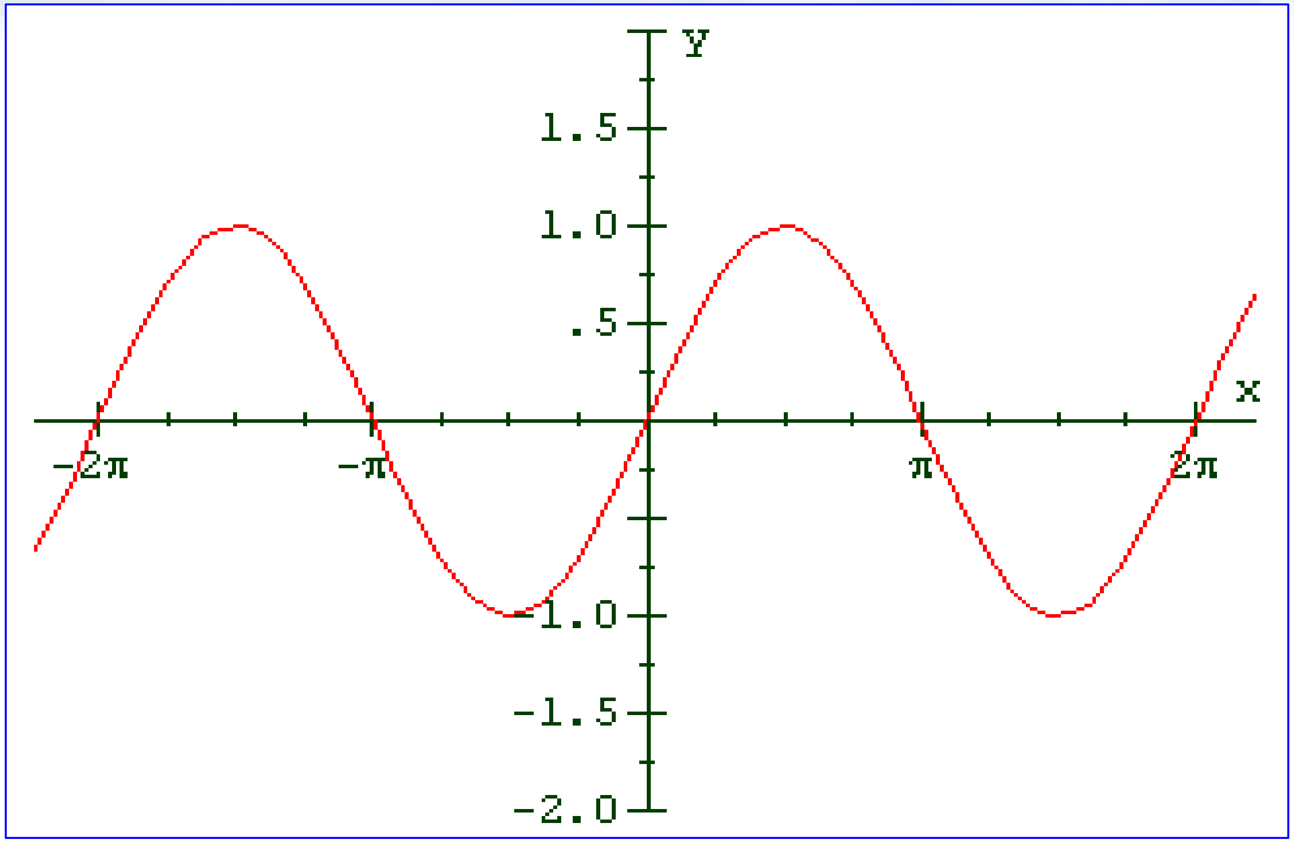
$$\sin\left(-\frac{5\pi}{2}\right) = -\sin\left(\frac{5\pi}{2}\right) = -\sin\left(2\pi + \frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$$

d. Using (5), (6), and Table 2, we have

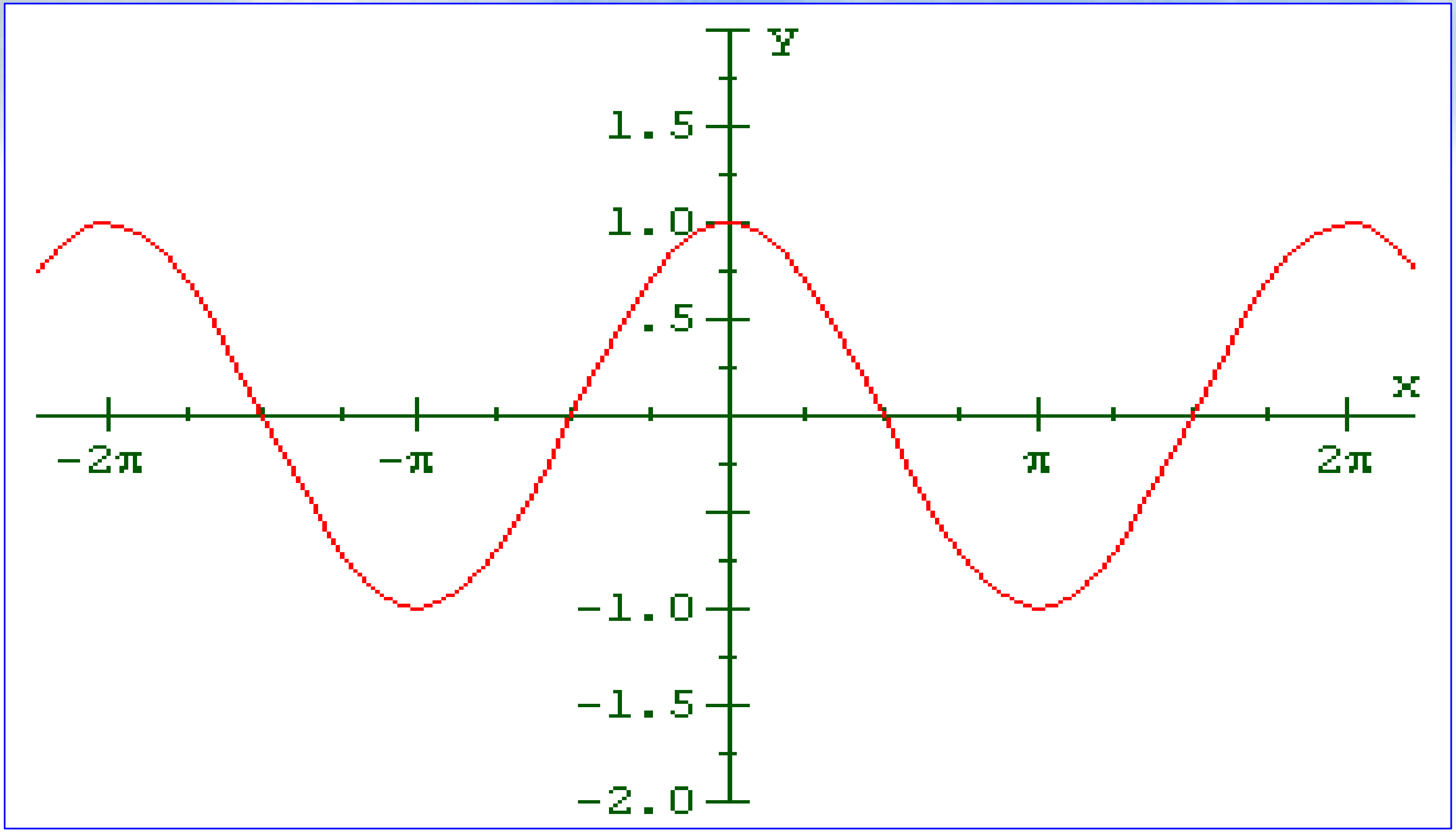
$$\cos\left(-\frac{11\pi}{4}\right) = \cos\frac{11\pi}{4} = \cos\left(2\pi + \frac{3\pi}{4}\right) = \cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$$

■

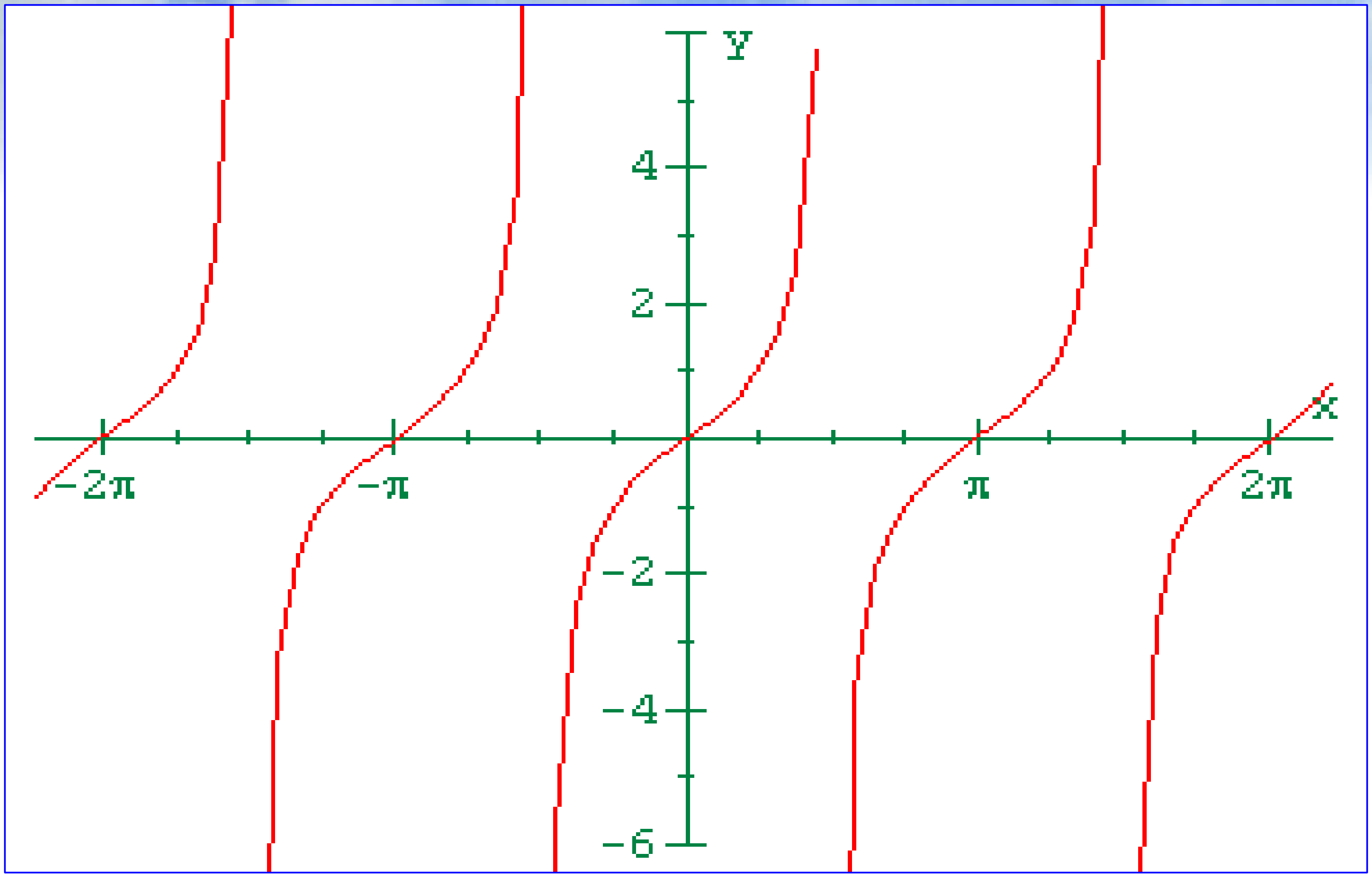
Graphs of Trigonometric Functions



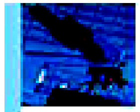
The graph of sin x



The graph of cos x



The graph of tan x



APPLIED EXAMPLE 2 Predator–Prey Population The population of owls (predators) in a certain region over a 2-year period is estimated to be

$$P_1(t) = 1000 + 100 \sin\left(\frac{\pi t}{12}\right)$$

in month t, and the population of mice (prey) in the same area at time t is given by

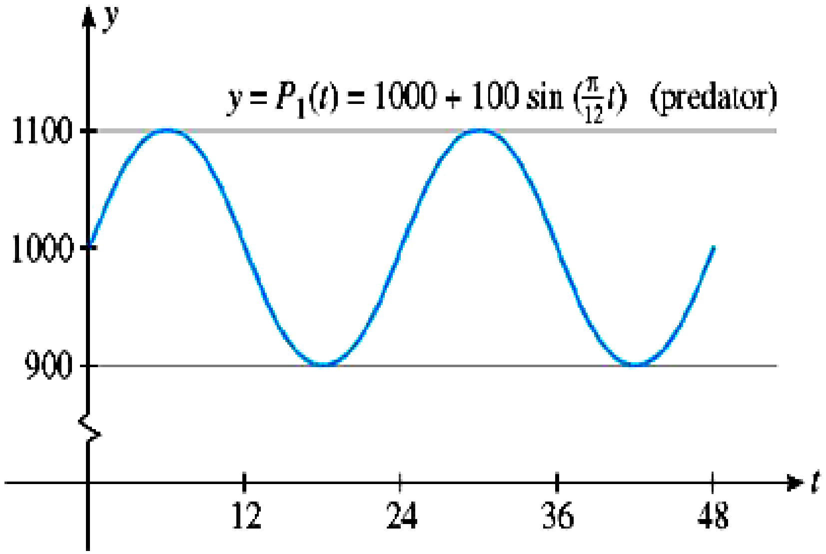
$$P_2(t) = 20{,}000 + 4000 \cos\left(\frac{\pi t}{12}\right)$$

Sketch the graphs of these two functions and explain the relationship between the sizes of the two populations.

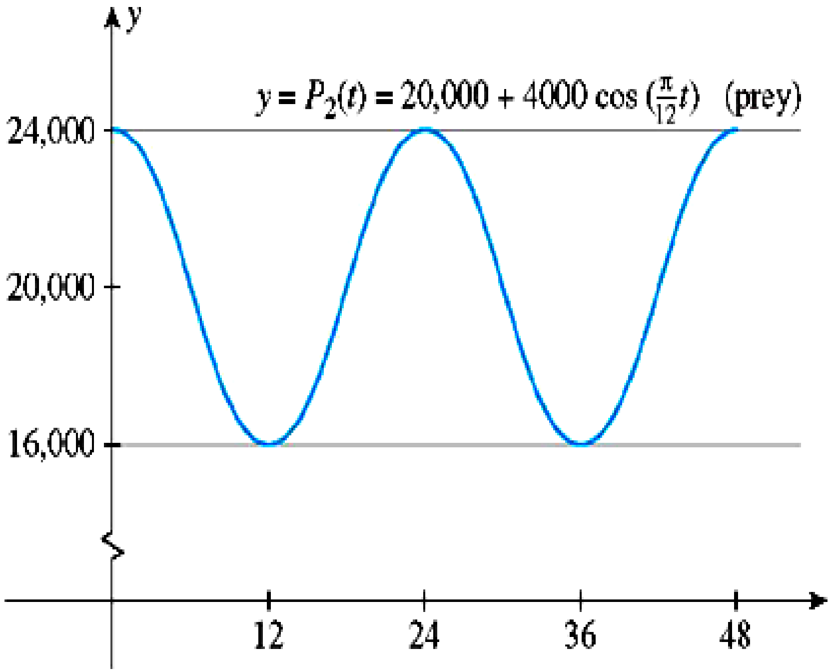
Solution We first observe that both of the given functions are periodic with period 24 (months). To see this, recall that both the sine and cosine functions are periodic with period 2π. Now the smallest value of $t > 0$ such that $\sin(\pi t/12) = 0$ is obtained by solving the equation

$$\frac{\pi t}{12} = 2\pi$$

giving $t = 24$ as the period of $\sin(\pi t/12)$. Since $P_1(t + 24) = P_1(t)$, we see that the function P_1 is periodic with period 24. Similarly, one verifies that the function P_2 is also periodic, with period 24, as asserted. Next, recall that both the sine and cosine functions oscillate between -1 and $+1$ so that $P_1(t)$ is seen to oscillate between $[1000 + 100(-1)]$, or 900, and $[1000 + 100(1)]$, or 1100, while $P_2(t)$ oscillates between $[20{,}000 + 4000(-1)]$, or 16,000, and $[20{,}000 + 4000(1)]$, or 24,000. Finally, plotting a few points on each graph for—say, $t = 0, 2, 3$, and so on—we obtain the graphs of the functions P_1 and P_2 as shown in Figure 15.



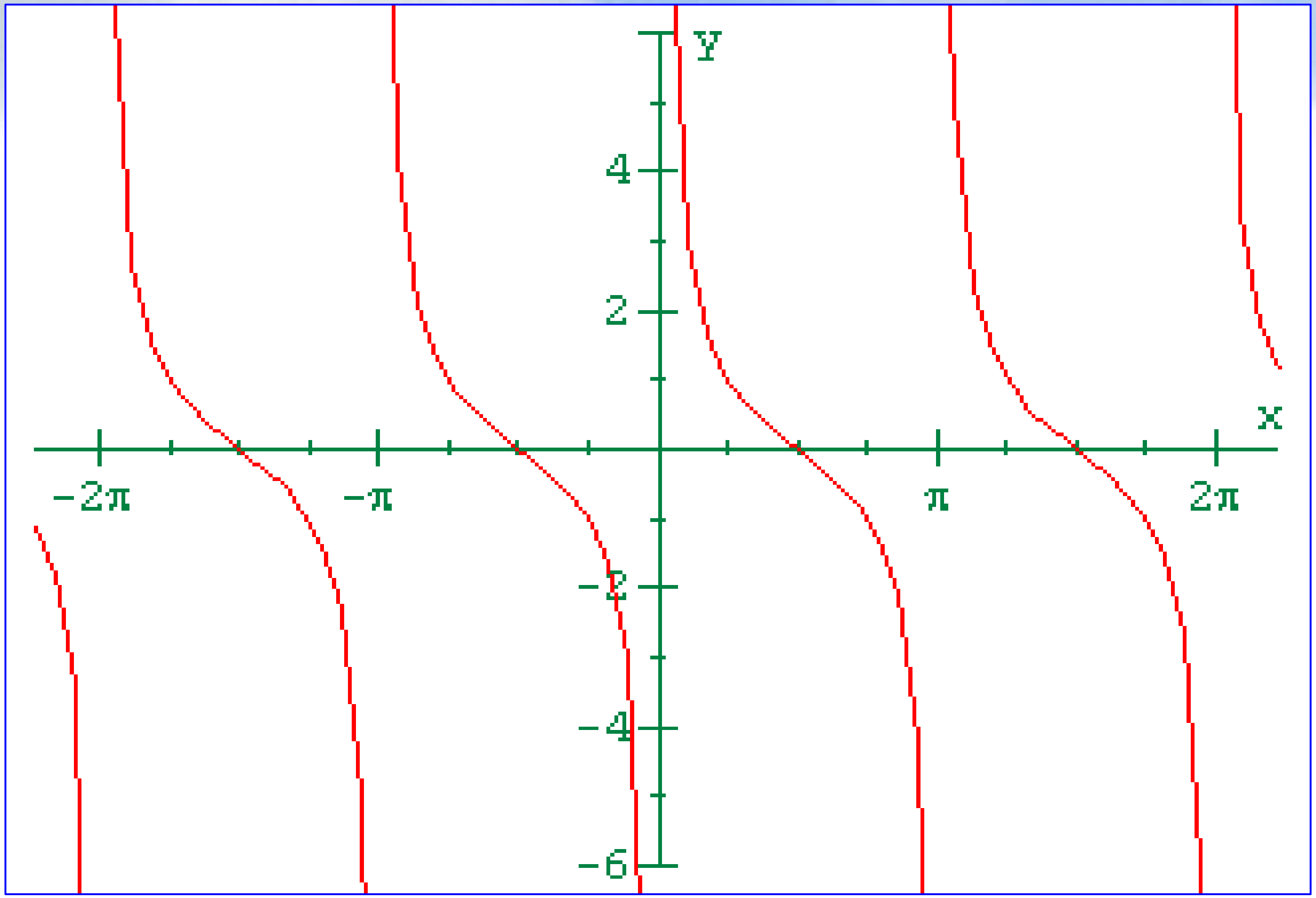
(a) The graph of the predator function $P_1(t)$



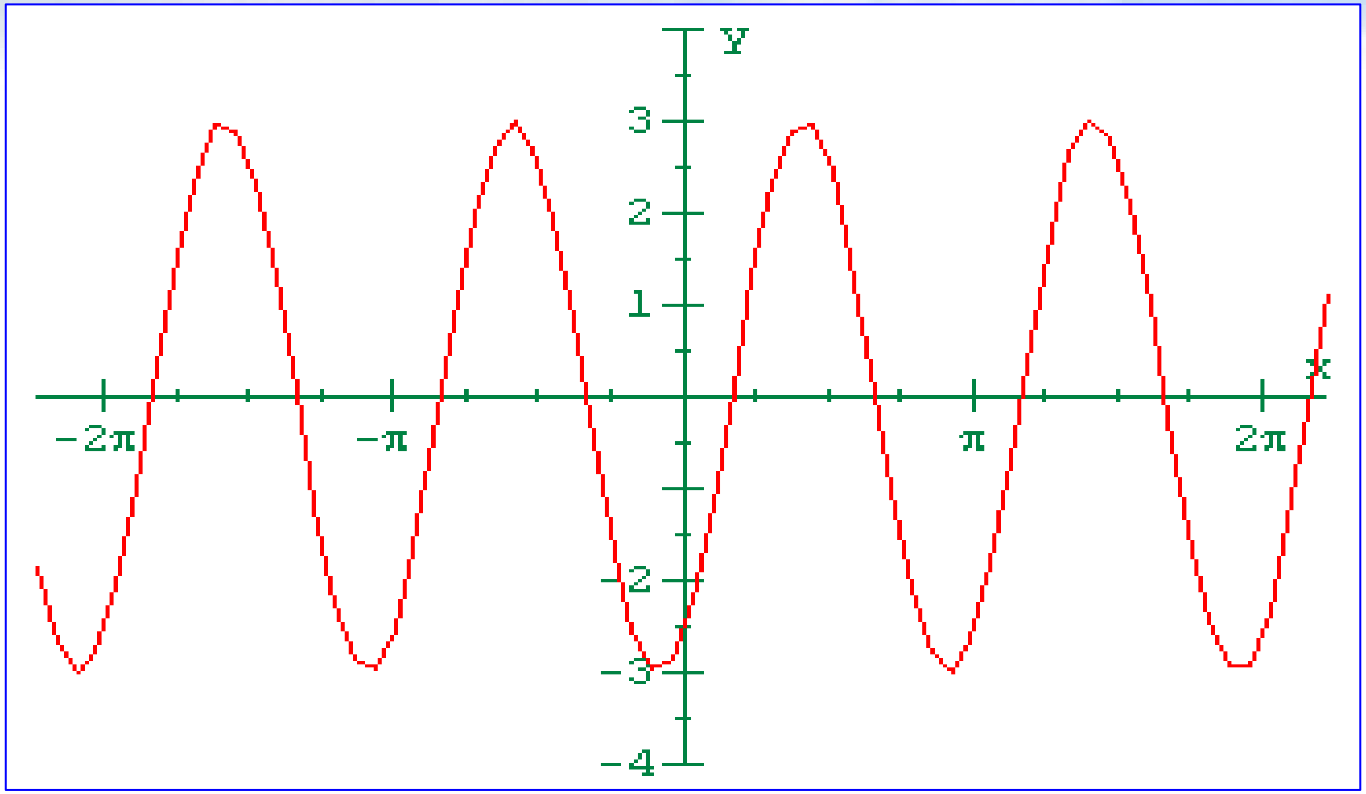
(b) The graph of the prey function $P_2(t)$

FIGURE 15

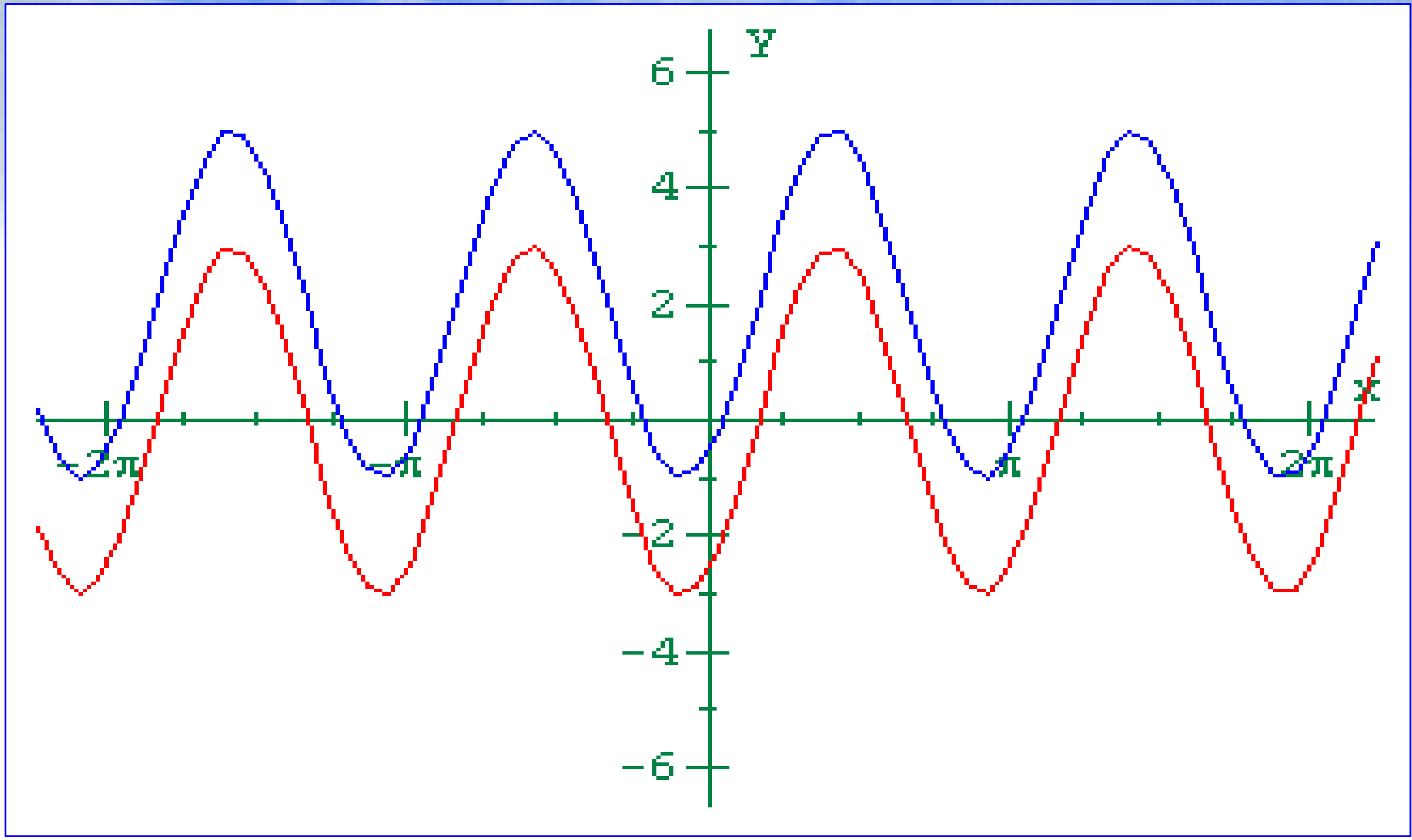
From the graphs, we see that at time $t = 0$ the predator population stands at 1000 owls. As it increases, the prey population decreases from 24,000 mice at that instant of time. Eventually, this decrease in the food supply causes the predator population to decrease, which in turn causes an increase in the prey population. But as the prey population increases, resulting in an increase in food supply, the predator population once again increases. The cycle is complete and starts all over again. ■



The graph of cot x



The graph of $y = 3\sin(2x - \frac{\pi}{3})$



The graph of $y = 3\sin(2x - \frac{\pi}{3})$ (red)

The graph of $y = 2 + 3\sin(2x - \frac{\pi}{3})$

Trigonometric Identities

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\cot^2 x + 1 = \csc^2 x$$

Half-Angle Formulas

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

Double-Angle Formulas

$$\sin 2A = 2\sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

Sum and Difference formulas

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

Cofunctions of Complementary Angles

$$\sin x = \cos\left(\frac{\pi}{2} - x\right)$$

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

Example

Verify the identity $\dfrac{\sec x - \cos x}{\tan x} = \sin x$

Solution

$$\frac{\sec x - \cos x}{\tan x} = \frac{\dfrac{1}{\cos x} - \cos x}{\dfrac{\sin x}{\cos x}} = \frac{1 - \cos^2 x}{\sin x}$$

$$= \frac{\sin^2 x}{\sin x} = \sin x$$

EXAMPLE 3 Verify the identity

$$\sin\theta(\csc\theta - \sin\theta) = \cos^2\theta$$

Solution We verify this identity by showing that the expression on the left side of the equation can be transformed into the expression on the right side. Thus,

$$\begin{aligned}\sin\theta(\csc\theta - \sin\theta) &= \sin\theta\csc\theta - \sin^2\theta \\ &= \sin\theta\frac{1}{\sin\theta} - \sin^2\theta \\ &= 1 - \sin^2\theta \\ &= \cos^2\theta\end{aligned}$$

12.3 Differentiation of Trigonometric Functions

Two Important Limits

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$$

Derivatives of the Sine and Generalized Sine Functions

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}\left[\sin f(x)\right] = \left[\cos f(x)\right] f'(x)$$

EXAMPLE 1 Differentiate each of the following functions:

a. $f(x) = x^2 \sin x$ **b.** $g(x) = \sin(2x + 1)$ **c.** $h(x) = (x + \sin x^2)^{10}$

Solution

a. Using the product rule followed by Rule 1a, we find

$$f'(x) = 2x \sin x + x^2 \frac{d}{dx}(\sin x)$$
$$= 2x \sin x + x^2 \cos x = x(2 \sin x + x \cos x)$$

b. Using Rule 1b, we find

$$g'(x) = [\cos(2x + 1)]\frac{d}{dx}(2x + 1) = 2 \cos(2x + 1)$$

c. We first use the general power rule followed by Rule 1b. We obtain

$$h'(x) = 10(x + \sin x^2)^9 \frac{d}{dx}(x + \sin x^2)$$
$$= 10(x + \sin x^2)^9 \left[1 + \cos x^2 \frac{d}{dx}(x^2)\right]$$
$$= 10(x + \sin x^2)^9(1 + 2x \cos x^2)$$

Derivatives of the Cosine and Generalized Cosine Functions

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}\left[\cos f(x)\right] = \left[-\sin f(x)\right] f'(x)$$

Example

Find the derivative of $f(x) = (x+10)(\sin x)$

Solution

Using the product rule, we have

$$\frac{df}{dx} = f'(x)$$

$$= (x+10)'(\sin x) + (x+1)(\sin x)'$$

$$= \sin x + (1+x)\cos x$$

Example

Find the derivative of the function

$$g(x) = \sin(3x^2 - 10x + 6)$$

Solution

Since $f(x) = 3x^2 - 10x + 6$

we find $f'(x) = 6x - 10$

then we have

$$g'(x) = (6x - 10)\cos(3x^2 - 10x + 6)$$

Example

Find the derivative of the function

$$g(x) = e^{\sin 5x - 2\cos 3x}$$

Solution

$$g'(x) = (e^{\sin 5x - 2\cos 3x})'$$

$$= \left(e^{\sin 5x - 2\cos 3x}\right)\left(\sin 5x - 2\cos 3x\right)'$$

$$= \left(e^{\sin 5x - 2\cos 3x}\right)\left(5\cos 5x + 6\sin 3x\right)$$

EXAMPLE 2 Find the derivative of each function:

a. $f(x) = \cos(2x^2 - 1)$ **b.** $g(x) = \sqrt{\cos 2x}$ **c.** $h(x) = e^{\sin 2x + \cos 3x}$

Solution

a. Using Rule 2b, we find

$$\begin{aligned} f'(x) &= -\sin(2x^2 - 1)\frac{d}{dx}(2x^2 - 1) \\ &= -[\sin(2x^2 - 1)]4x \\ &= -4x\sin(2x^2 - 1) \end{aligned}$$

b. We first rewrite $g(x)$ as $g(x) = (\cos 2x)^{1/2}$. Using the chain rule followed by Rule 2b, we find

$$\begin{aligned} g'(x) &= \frac{1}{2}(\cos 2x)^{-1/2}\frac{d}{dx}(\cos 2x) \\ &= \frac{1}{2}(\cos 2x)^{-1/2}(-\sin 2x)\frac{d}{dx}(2x) \\ &= \frac{1}{2}(\cos 2x)^{-1/2}(-\sin 2x)(2) \\ &= -\frac{\sin 2x}{\sqrt{\cos 2x}} \end{aligned}$$

c. Using the corollary to the chain rule for exponential functions, we find

$$h'(x) = e^{\sin 2x + \cos 3x} \cdot \frac{d}{dx}(\sin 2x + \cos 3x)$$

$$= e^{\sin 2x + \cos 3x}\left[(\cos 2x)\frac{d}{dx}(2x) - (\sin 3x)\frac{d}{dx}(3x)\right]$$

$$= (2\cos 2x - 3\sin 3x)e^{\sin 2x + \cos 3x}$$

Derivatives of Other Trigonometric Functions

Derivatives of Tangent Functions

$$\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x} = \sec^2 x$$

$$\frac{d}{dx}\left[\tan f(x)\right] = \left[\sec^2 f(x)\right] f'(x)$$

Derivatives of Cotangent Functions

$$\frac{d}{dx}(\cot x) = \frac{-1}{\sin^2 x} = -\csc^2 x$$

$$\frac{d}{dx}\left[\cot f(x)\right] = \left[-\csc^2 f(x)\right] f'(x)$$

Example

Differentiate the function $h(x)=\tan(x^2+3x-7)$

Solution

$f(x)=x^2+3x-7$ and so $f'(x)=2x+3$

Using the chain rule, we have

$$h'(x)=(2x+3)\sec^2(x^2+3x-7)$$

Derivatives of Secant Functions

$$\frac{d}{dx}(\sec x) = (\sec x)(\tan x)$$

$$\frac{d}{dx}\left[\sec f(x)\right] = \left[\sec f(x)\right]\left[\tan f(x)\right] f'(x)$$

Derivatives of Cosecant Functions

$$\frac{d}{dx}(\csc x) = -(\csc x)(\cot x)$$

$$\frac{d}{dx}\left[\csc f(x)\right] = -\left[\csc f(x)\right]\left[\cot f(x)\right] f'(x)$$

Example

Find the equation of the tangent line to the graph of the function $f(x) = \cot 2x$ at $\left(\frac{\pi}{4}, 0\right)$.

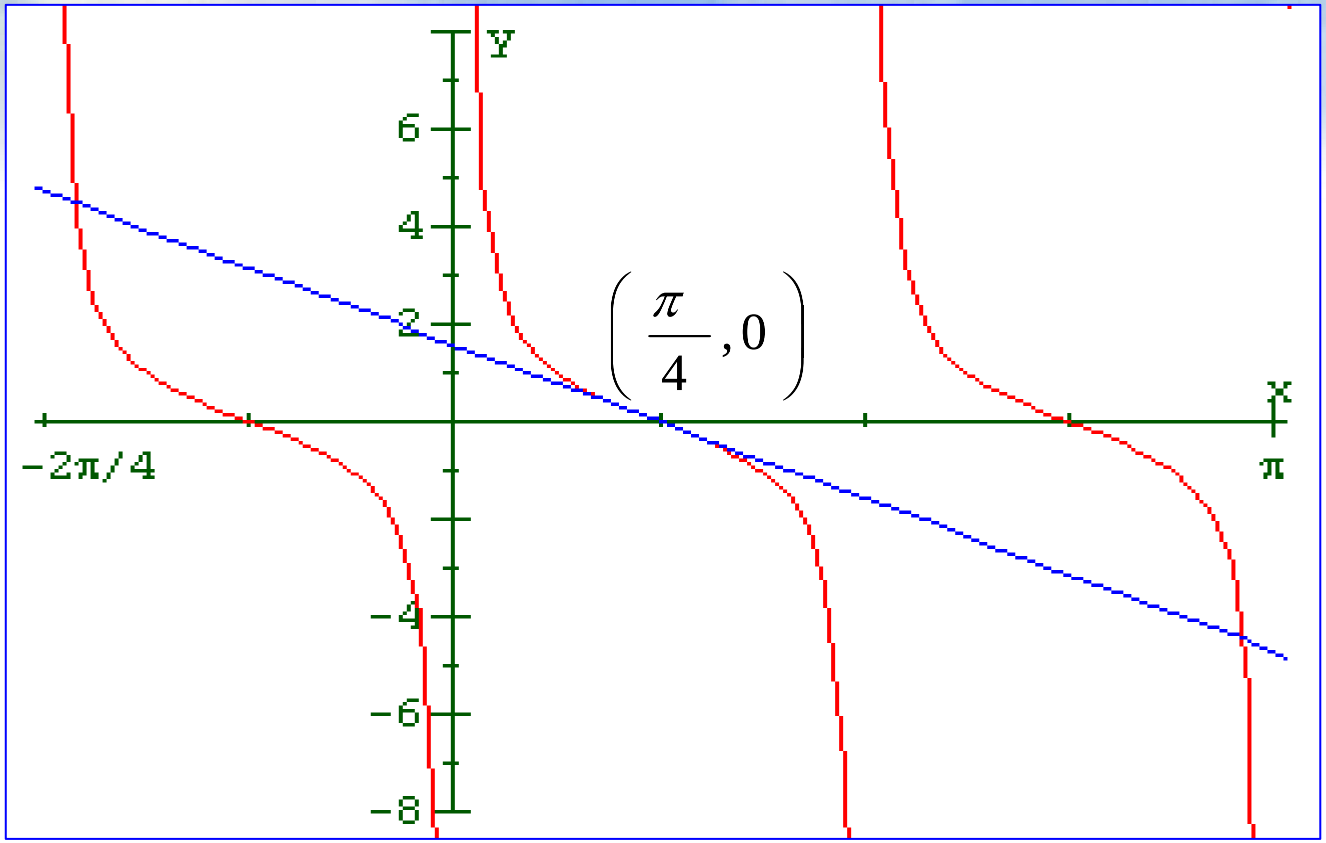
Solution

The slope of the tangent line at the given point is

$$f'(x)\Big|_{x=\frac{\pi}{4}} = (\cot 2x)'\Big|_{x=\frac{\pi}{4}} = \left(-\frac{2}{\sin^2 2x}\right)\Bigg|_{x=\frac{\pi}{4}} = -2$$

Therefore, the equation of the tangent line is

$$y = -2(x - \frac{\pi}{4}) \Rightarrow y = -2x + \frac{\pi}{2}$$



The graph of the function $f(x) = \cot 2x$

The graph of the tangent line $y = -2x + \frac{\pi}{2}$

EXAMPLE 3 Find an equation of the tangent line to the graph of the function $f(x) = \tan 2x$ at the point $(\pi/8, 1)$.

Solution The slope of the tangent line at any point on the graph of f is given by

$$f'(x) = 2\sec^2 2x$$

In particular, the slope of the tangent line at the point $(\pi/8, 1)$ is given by

$$f'\left(\frac{\pi}{8}\right) = 2\sec^2\frac{\pi}{4} = 2(\sqrt{2})^2 = 4$$

Therefore, a required equation is given by

$$y - 1 = 4\left(x - \frac{\pi}{8}\right)$$

$$y = 4x + \left(1 - \frac{\pi}{2}\right)$$

APPLIED EXAMPLE 4 **Predator–Prey Population** The owl population in a certain area is estimated to be

$$P_1(t) = 1000 + 100 \sin\left(\frac{\pi t}{12}\right)$$

in month t, and the mouse population in the same area at time t is given by

$$P_2(t) = 20{,}000 + 4000 \cos\left(\frac{\pi t}{12}\right)$$

Find the rate of change of each population when $t = 2$.

Solution The rate of change of the owl population at any time t is given by

$$\begin{aligned} P_1'(t) &= 100\left[\cos\left(\frac{\pi t}{12}\right)\right]\left(\frac{\pi}{12}\right) \\ &= \frac{25\pi}{3}\cos\left(\frac{\pi t}{12}\right) \end{aligned}$$

and the rate of change of the mouse population at any time t is given by

$$\begin{aligned} P_2'(t) &= 4000\left[-\sin\left(\frac{\pi t}{12}\right)\right]\left(\frac{\pi}{12}\right) \\ &= -\frac{1000\pi}{3}\sin\left(\frac{\pi t}{12}\right) \end{aligned}$$

In particular, when $t = 2$ the rate of change of the owl population is

$$P_1'(2) = \frac{25\pi}{3} \cos \frac{\pi}{6}$$
$$= \frac{25\pi}{3}\left(\frac{\sqrt{3}}{2}\right) \approx 22.7$$

That is, the predator population is increasing at the rate of approximately 22.7 owls per month, and the rate of change of the mouse population is

$$P_2'(2) = -\frac{1000\pi}{3} \sin \frac{\pi}{6}$$
$$= -\frac{1000\pi}{3}\left(\frac{1}{2}\right) \approx -523.6$$

That is, the prey population is decreasing at the rate of approximately 523.6 mice per month. ■

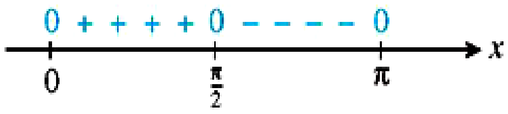


FIGURE 16
The sign diagram for f'

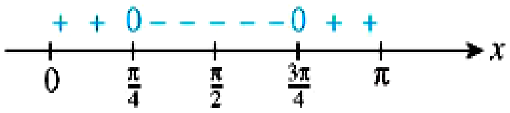


FIGURE 17
The sign diagram for f''

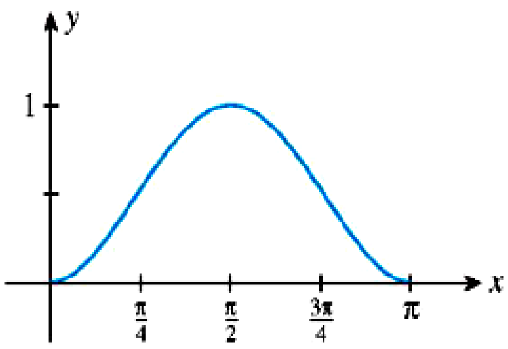


FIGURE 18
The graph of $y = \sin^2 x$

EXAMPLE 5 Sketch the graph of the function $y = f(x) = \sin^2 x$ on the interval $[0, \pi]$ by first obtaining the following information:

a. The intervals where f is increasing and where it is decreasing
b. The relative extrema of f
c. The concavity of f
d. The inflection points of f

Solution

a. $f'(x) = 2 \sin x \cos x = \sin 2x$. Setting $f'(x) = 0$ gives $x = 0, \pi/2$, or π. The sign diagram for f' (Figure 16) shows that f is increasing on $(0, \pi/2)$ and decreasing on $(\pi/2, \pi)$.

b. From the result of part (a) and the following table

x	0	$\frac{\pi}{2}$	π
$f(x)$	0	1	0

we see that the end points $x = 0$ and $x = \pi$ yield the absolute minimum of f, while the absolute maximum of f is 1.

c. We compute $f''(x) = 2 \cos 2x$. Setting $f''(x) = 0$ gives $2x = \pi/2$, or $3\pi/2$; that is, $x = \pi/4$, or $3\pi/4$. From the sign diagram for f'', shown in Figure 17, we conclude that f is concave upward on the intervals $(0, \pi/4)$ and $(3\pi/4, \pi)$ and concave downward on the interval $(\pi/4, 3\pi/4)$.

d. From the results of part (c), we see that $(\pi/4, 1/2)$ and $(3\pi/4, 1/2)$ are inflection points of f.

Finally, the graph of f is sketched in Figure 18. ■

APPLIED EXAMPLE 6 Restaurant Revenue The revenue of McMenamy's Fish Shanty, located at a popular summer resort, is approximately

$$R(t) = 2\left(5 - 4\cos\frac{\pi}{6}t\right) \qquad (0 \leq t \leq 12)$$

during the tth week ($t = 1$ corresponds to the first week of June), where R is measured in thousands of dollars. When is the weekly revenue increasing most rapidly?

Solution The revenue function R is increasing at the rate of

$$R'(t) = -8\left(-\sin\frac{\pi t}{6}\right)\left(\frac{\pi}{6}\right)$$

$$= \frac{4\pi}{3}\sin\frac{\pi t}{6}$$

thousand dollars per week. We want to maximize R'. Thus, we compute

$$R''(t) = \frac{4\pi}{3}\left(\cos\frac{\pi t}{6}\right)\left(\frac{\pi}{6}\right)$$

$$= \frac{2\pi^2}{9}\cos\frac{\pi t}{6}$$

Setting $R''(t) = 0$ gives

$$\cos\left(\frac{\pi t}{6}\right) = 0$$

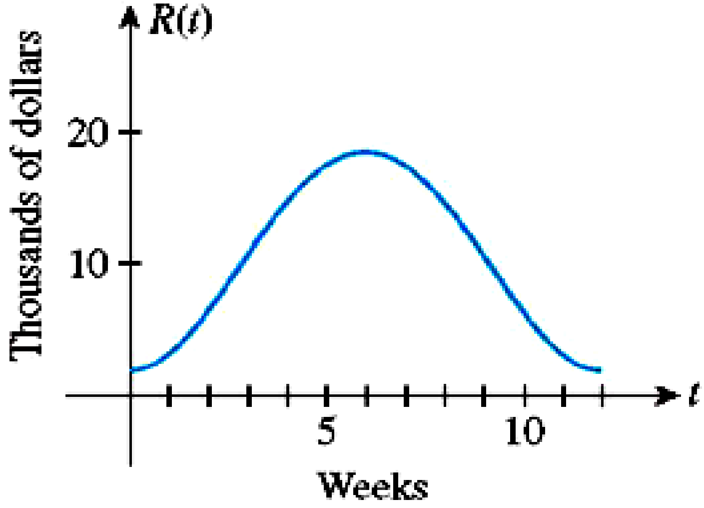


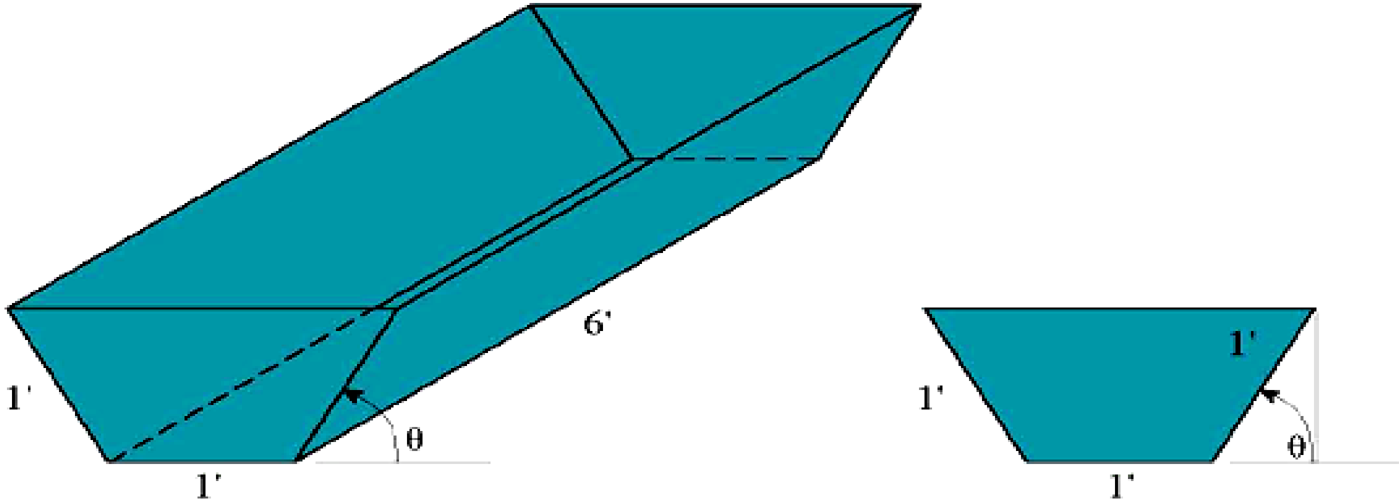
FIGURE 19

The graph of $R(t) = 2\left[5 - 4\cos\left(\frac{\pi}{6}t\right)\right]$

from which we see that $t = 3$ and $t = 9$ are critical numbers of $R'(t)$. Since $R'(t)$ is a continuous function on the closed interval [0, 12], the absolute maximum must occur at a critical number or at an endpoint of the interval. From the following table, we see that the weekly revenue is increasing most rapidly when $t = 3$—that is, in the third week of June (Figure 19).

t	0	3	9	12
$R'(t)$	0	$\frac{4\pi}{3}$	$-\frac{4\pi}{3}$	0

APPLIED EXAMPLE 7 Maximizing the Capacity of a Trough A trough with a trapezoidal cross section is to be constructed with a 1-foot base and sides that are 6 feet long (Figure 20). Find the angle of inclination θ that maximizes the capacity of the trough.



(a) We want to maximize the capacity of the trough with the given dimensions.

(b) A cross section of the trough has the shape of a trapezoid.

Solution The volume of the trough is the product of the area of its cross section and its length. Since the length is constant, it suffices to maximize the area of its cross section. But the cross section is a trapezoid with area given by one half the sum of the parallel sides times its height, or

$$\begin{aligned} A &= \frac{1}{2}[1 + (1 + 2\cos\theta)]\sin\theta \\ &= (1 + \cos\theta)\sin\theta \end{aligned}$$

To find the absolute maximum of the continuous function A over the closed interval $[0, \pi/2]$, we first compute the derivative of A, obtaining

$$\begin{aligned} A' &= -\sin^2\theta + (1 + \cos\theta)\cos\theta \\ &= -\sin^2\theta + \cos\theta + \cos^2\theta \\ &= (\cos^2\theta - 1) + \cos\theta + \cos^2\theta \qquad \text{Use the identity } \sin^2\theta + \cos^2\theta = 1. \\ &= 2\cos^2\theta + \cos\theta - 1 \\ &= (2\cos\theta - 1)(\cos\theta + 1) \end{aligned}$$

Setting $A' = 0$ gives $\cos\theta = 1/2$, or $\cos\theta = -1$; that is, $\theta = \pi/3$ [the other values of θ lie outside the interval $(0, \pi/2)$]. Evaluating the function A at the critical number $\theta = \pi/3$ of A and at the endpoints of the interval, we have

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$
A	0	$\frac{3\sqrt{3}}{4}$	1

from which we see that the volume of the trough is maximized when $\theta = \pi/3$, or $60°$.

12.4 Integration of Trigonometric Functions

Formulas

$$\int (\sin x)\,dx = -\cos x + C$$

$$\int (\cos x)\,dx = \sin x + C$$

$$\int (\sec^2 x)\,dx = \tan x + C$$

$$\int (\csc^2 x)\,dx = -\cot x + C$$

$$\int (\sec x \tan x)\,dx = \sec x + C$$

$$\int (\csc x \cot x)\,dx = -\csc x + C$$

Example

Evaluate $\int \left(\frac{\cos x}{5 + \sin x} \right) dx$

Solution

Let $u = 5 + \sin x$, then $du = \cos x \, dx$

We have

$$\int \left(\frac{\cos x}{5 + \sin x} \right) dx = \int \frac{du}{u}$$

$$= \ln |u| + c$$

$$= \ln |5 + \sin x| + C$$

Example

Find the area under the curve of $f(t) = \sin t$

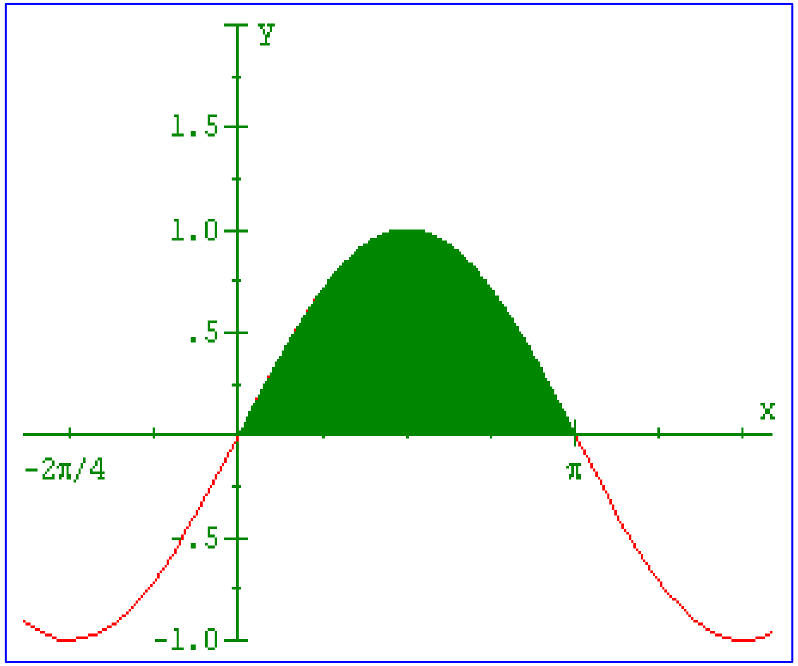
from $t = 0$ to $t = \pi$.

Solution

The required area is given by

$$A = \int_0^{\pi} (\sin t)\, dt$$

$$= (-\cos t)\Big|_0^{\pi} = 2$$

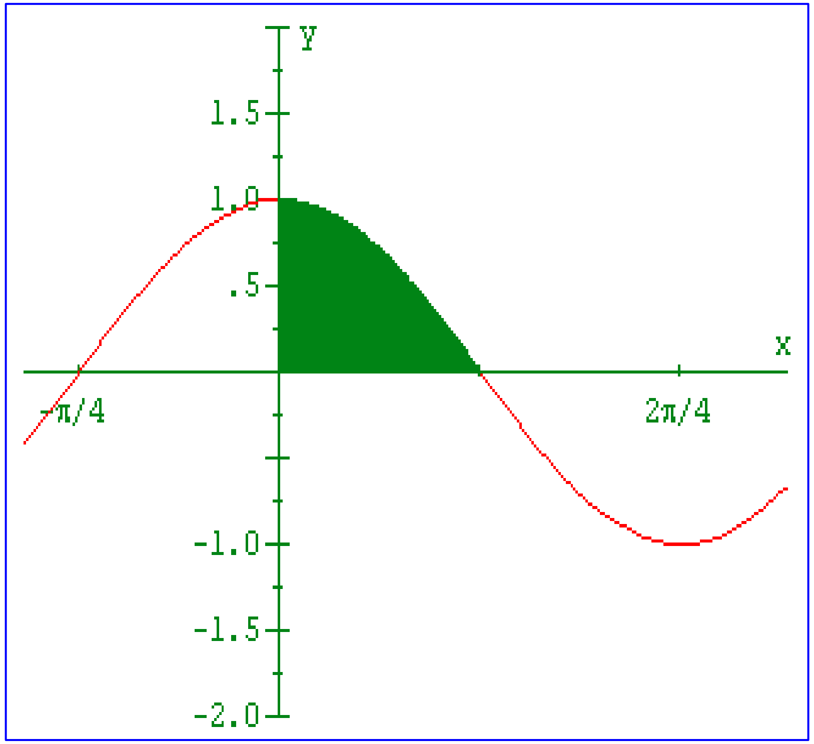


Example

Find the area under the curve of $f(t) = \cos 2t$ from $t = 0$ to $t = \frac{\pi}{4}$.

Solution

The required area is given by



$$A = \int_0^{\pi/4} (\cos 2t)\, dt$$

$$= \left(\frac{1}{2} \sin 2t \right) \Big|_0^{\pi/4} = \frac{1}{2}$$

EXAMPLE 1 Evaluate $\int \cos 3x \, dx$.

Solution Let's put $u = 3x$, so that $du = 3\,dx$ and $dx = \frac{1}{3}\,du$. Then

$$\int \cos 3x \, dx = \frac{1}{3}\int \cos u \, du$$

$$= \frac{1}{3}\sin u + C \qquad \text{Use Rule 2.}$$

$$= \frac{1}{3}\sin 3x + C$$

■

EXAMPLE 2 Evaluate $\int \sec(2x+1)\tan(2x+1)\,dx$.

Solution Put $u = 2x + 1$ so that $du = 2\,dx$, or $dx = \frac{1}{2}\,du$. Then

$$\int \sec(2x+1)\tan(2x+1)\,dx = \frac{1}{2}\int \sec u \tan u\,du$$

$$= \frac{1}{2}\sec u + C$$

$$= \frac{1}{2}\sec(2x+1) + C$$

EXAMPLE 3 Evaluate $\displaystyle\int \frac{\sin x\, dx}{1 + \cos x}$.

Solution Put $u = 1 + \cos x$ so that $du = -\sin x\, dx$, or $\sin x\, dx = -du$. Then

$$\int \frac{\sin x\, dx}{1 + \cos x} = -\int \frac{du}{u}$$

$$= -\ln|u| + C$$

$$= -\ln|1 + \cos x| + C$$

EXAMPLE 4 Evaluate $\int_0^{\pi/2} x \cos 2x \, dx$.

Solution We first integrate the indefinite integral

$$\int x \cos 2x \, dx$$

by parts with

$$u = x \qquad \text{and} \qquad dv = \cos 2x \, dx$$

so that

$$du = dx \qquad \text{and} \qquad v = \frac{1}{2} \sin 2x$$

Therefore,

$$\begin{aligned} \int x \cos 2x \, dx &= uv - \int v \, du \\ &= \frac{1}{2} x \sin 2x - \frac{1}{2} \int \sin 2x \, dx \\ &= \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + C \end{aligned}$$

We have used the method of substitution to evaluate the integral on the right.
Therefore,

$$\int_0^{\pi/2} x\cos 2x\,dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\Big|_0^{\pi/2}$$

$$= \left[\frac{1}{2}\left(\frac{\pi}{2}\right)\sin\pi + \frac{1}{4}\cos\pi\right] - \left(0 + \frac{1}{4}\cos 0\right)$$

$$= -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

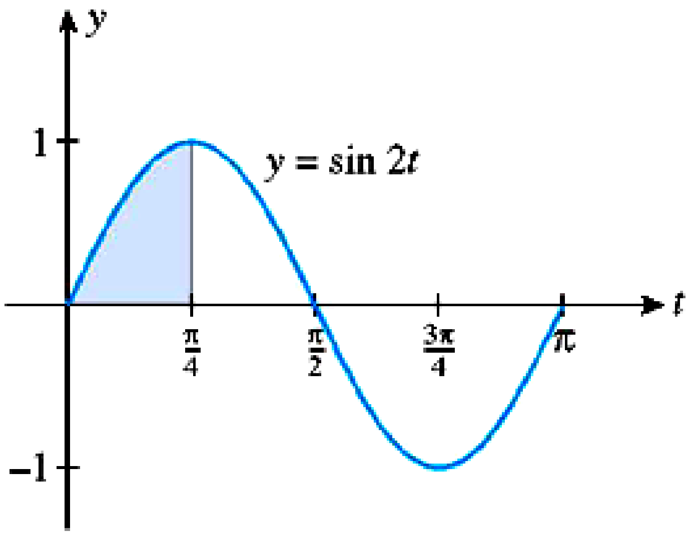


FIGURE 21
The area under the curve $y = \sin 2t$ from $t = 0$ to $t = \frac{\pi}{4}$

EXAMPLE 5 Find the area under the curve of $y = \sin 2t$ from $t = 0$ to $t = \pi/4$.

Solution The required area, shown in Figure 21, is given by

$$\int_0^{\pi/4} \sin 2t\, dt = -\frac{1}{2}\cos 2t \Big|_0^{\pi/4} = -\frac{1}{2}\cos\frac{\pi}{2} + \frac{1}{2}\cos 0 = \frac{1}{2}$$

or $\frac{1}{2}$ square unit. ■

APPLIED EXAMPLE 6 **Stock Prices** The weekly closing price of HAL Corporation stock in week t is approximated by the rule

$$f(t) = 30 + t \sin \frac{\pi}{6} t \qquad (0 \le t \le 15)$$

where $f(t)$ is the price (in dollars) per share. Find the average weekly closing price of the stock over the 15-week period.

Solution The average weekly closing price of the stock over the 15-week period in question is given by

$$\begin{aligned} A &= \frac{1}{15-0} \int_0^{15} \left(30 + t \sin \frac{\pi}{6} t\right) dt \\ &= \frac{1}{15} \int_0^{15} 30\, dt + \frac{1}{15} \int_0^{15} t \sin \frac{\pi}{6} t\, dt \\ &= 30 + \frac{1}{15} \int_0^{15} t \sin \frac{\pi}{6} t\, dt \end{aligned}$$

Integrating by parts with

$$u = t \qquad \text{and} \qquad dv = \sin \frac{\pi}{6} t\, dt$$

so that

$$du = dt \qquad \text{and} \qquad v = -\frac{6}{\pi}\cos\frac{\pi}{6}t$$

we have

$$\begin{aligned} A &= 30 + \frac{1}{15}\left(-\frac{6}{\pi}t\cos\frac{\pi}{6}t\Big|_0^{15} + \frac{6}{\pi}\int_0^{15}\cos\frac{\pi}{6}t\,dt\right) \\ &= 30 + \frac{1}{15}\left[-\left(\frac{6}{\pi}\right)(15)\cos\frac{15\pi}{6} + \frac{6}{\pi}(0)\cos 0 + \left(\frac{6}{\pi}\right)^2\sin\frac{\pi}{6}t\Big|_0^{15}\right] \\ &= 30 + \frac{1}{15}\left[-\left(\frac{6}{\pi}\right)(15)(0) + \left(\frac{6}{\pi}\right)^2\sin\frac{15\pi}{6} - \left(\frac{6}{\pi}\right)^2\sin 0\right] \\ &= 30 + \frac{1}{15}\left(\frac{6}{\pi}\right)^2(1) \approx 30.24 \end{aligned}$$

or approximately $30.24 per share. ■